Some notes on integration:

$$\int e^x dx = e^x + C$$

$$\int e^{2x} dx = \frac{1}{2} e^{2x} + C$$ because

$$u = 2x \Rightarrow du = 2dx$$

$$\frac{1}{2}\int e^{2x} 2dx$$

$$-\cancel{\frac{1}{2x}}\int -\cancel{2x} e^{-x^2} dx \qquad u = -x^2 \Rightarrow du = -2x\,\boxed{dx}$$

$$\frac{dy}{dx} = (1+y^2) e^{-x^2}$$

$$\int_0^x \frac{dy}{1+y^2} = \int_0^x e^{-\cancel{x}t^2} d\cancel{x}t$$

$$\int \frac{du}{u}$$

$$\tan^{-1} y(t)\Big|_0^x = \int_0^x e^{-t^2} dt$$

$$\tan^{-1} y = \int_0^x e^{-t^2} dt$$

$3y'' + 4y' + y = 0$ $\qquad$ $2x^2y'' + 5xy' - 7y = 0$

4.1 - Preliminary Theory–Linear Equations

Definition: The equations

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \qquad (1) \quad \text{and}$$

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (2)$$

are nth-order, linear differential equations. (1) is **homogeneous**, and if $g(x) \not\equiv 0$, then (2) is **nonhomogeneous**.

Definition: The points $y(x_0) = y_0, y'(x_0) = y_1, \cdots, y^{(n-1)}(x_0) = y_{n-1}$ are **initial conditions**. An **initial-value problem** is (1) or (2), together with initial conditions. In the event that we have information such as $y(a) = y_0,\ y(b) = y_1$, then we have a **boundary-value problem**.

$y(a)$ $\quad$ $y'(b)$ $\quad$ $y'(a)$ $\quad$ $y''(b)$

Theorem: Existence of a Unique Solution

Let $a_n(x), a_{n-1}(x), \ldots, a_1(x)$ and $g(x)$ be continuous on an interval I and let $a_n(x) \neq 0$ for every x in this interval. If $x = x_0$ is any point in this interval, then a solution $y(x)$ of the initial-value problem (1) exists on the interval and is unique.

These two important criteria will be assumed henceforth in the statement of theorems about linear equations. A BVP can have zero, one, or infinitely many solutions

Example: The given two-parameter family is a solution of the indicated differential equation on the interval $(-\infty, \infty)$. Determine whether a member of the family can be found that satisfies the boundary conditions.

$y = c_1x^2 + c_2x^4 + 3; \quad x^2y'' - 5xy' + 8y = 24$

(a) $y(-1) = 0, \quad y(1) = 4$ $\qquad$ (b) $y(0) = 1, \quad y(1) = 2$

(c) $y(0) = 3, \quad y(1) = 0$ $\qquad$ (d) $y(1) = 3, \quad y(2) = 15$

a) $y(-1) = 0 \Rightarrow c_1 + c_2 + 3 = 0$

$y(1) = 4 \Rightarrow c_1 + c_2 + 3 = 4$ Not possible.

No solution.

b) $y(0) = 1 \Rightarrow 3 = 1$ stop. No.

c) $y(0) = 3 \Rightarrow 3 = 3$

$y(1) = 0 \Rightarrow c_1 + c_2 + 3 = 0$

has a solution as long as $c_1 = -3 - c_2$

infinitely many solutions.

Solution: $(-3-c_2)x^2 + c_2 x^4 + 3 = 0$

d) Will have one solution

$\sin^2 x, \cos 2x, 1$ $\quad c_1 \sin^2 x + c_2 \cos 2x + c_3 = 0$

$y_1 = e^{2x}, y_2 = e^{3x} \rightarrow c_1 e^{2x} + c_2 e^{3x} = 0$

Definition: A set of functions $f_1(x), f_2(x), \dots, f_n(x)$ is said to be **linearly dependent** on an interval I if there exist constants $c_1, c_2, \dots, c_n$, not all zero, such that $c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$ for every x in the interval. If the set of functions is not linearly dependent on the interval, it is said to be **linearly independent**

Definition: Suppose each of the functions $f_1(x), f_2(x), \dots, f_n(x)$ possesses at least $n-1$ derivatives. The determinant

$$W(f_1, f_2, \dots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix},$$

where the primes denote derivatives, is called the **Wronskian** of the functions

Theorem: Criterion for Linearly Independent Solutions

Let $y_1, y_2, \ldots, y_n$ be n solutions of the homogeneous nth-order differential equation (1) on an interval I. Then the set of solutions is **linearly independent** on I if and only if $W(y_1, y_2, \ldots, y_n) \neq 0$ for every x in the interval.

Note: The Wronskian is either always or never zero on an interval.

Expansion by minors

Example: Determine whether the given set of functions is linearly independent on the interval $(-\infty, \infty)$.

$f_1(x) = \cos 2x,\ f_2(x) = 1,\ f_3(x) = \cos^2 x$

Side note: $\begin{vmatrix} \overset{+}{a} & \overset{-}{b} & \overset{+}{c} \\ \overset{-}{d} & \overset{+}{e} & \overset{-}{f} \\ \overset{+}{g} & \overset{-}{h} & \overset{+}{i} \end{vmatrix}$

$+a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

$a(ei - hf) - b(di - gf) + c(dh - ge)$

$$W(f_1, f_2, f_3) = \begin{vmatrix} \overset{+}{\cos 2x} & \overset{-}{1} & \overset{+}{\cos^2 x} \\ -2\sin 2x & 0 & \cancel{-2\cos x \sin x}\ -\sin 2x \\ -4\cos 2x & 0 & -2\cos 2x \end{vmatrix}$$

$$-1\begin{vmatrix} -2\sin 2x & -\sin 2x \\ -4\cos 2x & -2\cos 2x \end{vmatrix}$$

$$= -(4\cos 2x \sin 2x - 4\cos 2x \sin 2x)$$

$$= 0 \quad \forall x$$

Not independent.

Theorem: Superposition Principle–Homogeneous Equations

Let $y_1, y_2, \ldots, y_k$, be solutions of the homogeneous nth-order differential equation (1) on an interval I. Then the linear combination $c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x)$, where the c_i, $i = 1, 2, \ldots, k$ are arbitrary constants, is also a solution on the interval.

$y = e^{3x} + e^{-5x} \Rightarrow y' = 3e^{3x} - 5e^{-5x},\ y'' = 9e^{3x} + 25e^{-5x}$

Ex: $y_1 = e^{3x}$ and $y_2 = e^{-5x}$ are solutions to $y'' + 2y' - 15y = 0$.

check: $y_1' = 3e^{3x}$, $y_1'' = 9e^{3x}$: $9e^{3x} + 6e^{3x} - 15e^{3x} = 0$ ✓

so is $y = c_1 e^{3x} + c_2 e^{5x}$ for any $c_1, c_2 \in \mathbb{R}$

Definition: Any set $y_1, y_2, \ldots, y_n$, of n of linearly independent solutions of the homogeneous nth-order differential equation (1) on an interval I is said to be a **fundamental set of solutions** on the interval.

Theorem: General Solution–Homogeneous Equations

Let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the homogeneous linear nth-order differential equation (1) on an interval I. Then the **general solution** of the equation on the interval is $y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$.

Example: Verify that the given functions form a fundamental set of solutions of the differential equation on the indicated interval. Form the general solution.

$x^2 y'' + xy' + y = 0$; $\cos(\ln x), \sin(\ln x), (0, \infty)$

Independent?

$$W = \begin{vmatrix} \cos(\ln x) & \sin(\ln x) \\ \frac{-\sin(\ln x)}{x} & \frac{\cos(\ln x)}{x} \end{vmatrix}$$

$$= \frac{\cos^2(\ln x) + \sin^2(\ln x)}{x} = \frac{1}{x} \neq 0$$

Independent ✓

Solutions? $y_1 = \cos(\ln x) \Rightarrow y_1' = -\frac{\sin(\ln x)}{x}$

$$y_1'' = \frac{\frac{-\cos(\ln x)}{x} x + \sin(\ln x)}{x^2} = \frac{-\cos(\ln x) + \sin(\ln x)}{x^2}$$

$x^2y'' + xy' + y \overset{?}{=} 0$:

$-\cos(\ln x) + \sin(\ln x) - \sin(\ln x) + \cos(\ln x) = 0$ ✓

Likewise for y_2.

$y = c_1\cos(\ln x) + c_2\sin(\ln x)$

Theorem: **Existence of a Fundamental Set**

There exists a fundamental set of solutions for the homogenous linear nth-order differential equation (1) on an interval I.

Definition: Any function y_p, free of arbitrary parameters, that satisfies (2) is said to be a **particular solution** of the equation.

Theorem: **General Solution–Nonhomogeneous Equations**

Let y_p be any particular solution of the nonhomogeneous linear nth-order differential equation (2) on an interval I and let $y_1, y_2, \dots, y_n$ be a fundamental set of solutions of the associated homogeneous differential equation (1) on I. Then the **general solution** of the equation on the interval is

$y = \boxed{c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)} + \boxed{y_p(x)}$, where the c_i, $i = 1, 2, \dots, n$ are arbitrary constants.

From homog. eqn. — because an eqn is nonhomog

$y = y_c + y_p$

Theorem: **Superposition Principle–Nonhomogeneous Equations**

Let $y_{p_1}, y_{p_2}, \dots, y_{p_k}$ be k particular solutions of the nonhomogeneous linear nth-order differential equation (2) on an interval I corresponding, in turn, to k distinct functions $g_1, g_2, \dots, g_k$. Then $y_p(x) = y_{p_1}(x) + y_{p_2}(x) + \dots + y_{p_k}(x)$ is a particular solution of

$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g_1(x) + g_2(x) + \dots + g_k(x)$

Example: (a) Verify that $y_{p_1} = 3e^{2x}$ and $y_{p_2} = x^2 + 3x$, respectively are particular solutions of

① $y'' - 6y' + 5y = -9e^{2x}$ and

② $y'' - 6y' + 5y = 5x^2 + 3x - 16.$

$y_{p_1} \to$ ① : $y_{p_1}' = 6e^{2x}$, $y_{p_1}'' = 12e^{2x}$

$12e^{2x} - 6(6e^{2x}) + 5(3e^{2x})$

$= (12 - 36 + 15)e^{2x} = -9e^{2x}$ ✓

Likewise for y_{p_2}.

(b) Use part (a) to find particular solutions of

$y'' - 6y' + 5y = 5x^2 + 3x - 16 - 9e^{2x}.$

$y_p = x^2 + 3x + 3e^{2x}$

Extending: It happens that $y_1 = e^{5x}$ and $y_2 = e^x$ are solutions to $y'' - 6y' + 5y = 0$ (homog.)

So $y_c = c_1 e^{5x} + c_2 e^x$ is the general solution

$y = \underbrace{y_c}_{} + \underbrace{y_p}_{} \Rightarrow y = \underbrace{c_1 e^{5x} + c_2 e^x}_{4.3} + \underbrace{x^2 + 3x + 3e^{2x}}_{4.4}$